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A Characterization of Valuation Domains via m-Canonical Ideals#

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ABSTRACT

A nonzero ideal I of an intergral domain R is said to be an m-canonical ideal of R if $(I:(I:J)) = J$ for every nonzero ideal J of R. In this paper, we show that if a quasi-local integral domain (R, M) admits a proper m-canonical ideal I of R, then the following statements are equivalent:

(1) R is a valuation domain.
(2) I is a divided m-canonical ideal of R.
(3) $cM = I$ for some nonzero $c \in R$.
(4) $(I:M)$ is a principal ideal of R.
(5) $(I:M)$ is an invertible ideal of R.
(6) R is an integrally closed domain and $(I:M)$ is a finitely generated of R.
(7) $(M:M) = R$ and $(I:M)$ is a finitely generated of R.
(8) If $J = (I:M)$, then J is a finitely generated of R and $(J:J) = R$.

Among the many results in this paper, we show that an integral domain R is a valuation domain if and only if R admits a divided proper m-canonical ideal,

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iff R is a root closed domain which admits a strongly primary proper m-canonical ideal, also we show that an integral domain R is a one-dimensional valuation domain if and only if R is a completely integrally closed domain which admits a powerful proper m-canonical ideal of R.

Key Words: m-Canonical ideals; Valuation domains; Divided ideals; Prüfer domain.

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1. INTRODUCTION

Throughout this paper, R denotes a commutative integral domain with identity $1 \neq 0$ having quotient field K and (R, M) denotes a quasi-local domain with maximal ideal M. If J and L are fractional ideals of R then $(J : L) = \{x \in K | xL \subset J\}$ and $J^{-1} = (R : J)$. Recall that an ideal I of R is called *divisorial* if $(R : (R : I)) = I$. We recall from Heinzer et al. (1998) that a nonzero ideal I of R is said to be m-*canonical* if $(I : (I : J)) = J$ for every nonzero ideal J of R. This type of ideals has been studied extensively in Heinzer et al. (1998) and Barucci et al. (2001). Other related studies can be found in Bass (1963), Bazonni and Salce (1996), Heinzer (1968), Herzog and Kunz (1971), Houston and Zafrullah (1988), Matils (1968), Nagata (1962), Olberding (1998), and Vasconcelos (1974). We say that an ideal I of R is *proper* if $I \neq \{0\}$ and $I \neq R$. In this paper, we show (Corollary 2.15) that if a quasi-local integral domain (R, M) admits a proper m-canonical ideal I of R, then the following statements are equivalent:

(1) R is a valuation domain.
(2) I is a divided m-canonical ideal of R.
(3) $cM = I$ for some nonzero $c \in R$.
(4) $(I : M)$ is a principal ideal of R.
(5) $(I : M)$ is an invertible ideal of R.
(6) R is an integrally closed domain and $(I : M)$ is finitely generated of R.
(7) $(M : M) = R$ and $(I : M)$ is a finitely generated of R.
(8) If $J = (I : M)$, then J is finitely generated of R and $(J : J) = R$.

Recall that a proper ideal I of an integral domain R is said to be *divided* in the sense of Dobbs (1976) and Badawi (1999) if $I \subset (c)$ for every $c \in R \backslash I$. We show (Corollary 2.5 and Theorem 3.3) that an integral domain R is a valuation domain if and only if R admits a divided proper m-canonical ideal of R. We recall from Badawi and Houston (2002) that an ideal I of R is said to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. We show (Corollary 3.4) that an integral domain R is a valuation domain if and only if R is a root closed domain which admits a strongly primary proper m-canonical ideal; also recall from Badawi and Houston (2002) that an ideal I of R is called *powerful* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in R$ or $y \in R$. We show (Corollary 3.5) that an integral domain R is a one-dimensional valuation domain if and only if R is a completely integrally closed domain which admits a powerful proper m-canonical ideal of R. We recall that R is called an *h-local domain* if each nonzero ideal of R is contained in only finitely many maximal ideals of R and each nonzero prime ideal of R is contained in a unique maximal ideal of R. Suppose that an integrally closed domain R admits a proper m-canonical I such that $(I : M)$ is a finitely generated ideal of R for every maximal ideal M of R containing I. Then we show (Theorem 3.6) that R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated. We show (Proposition 3.8) that if an integrally closed domain R admits a proper m-canonical ideal I, then the following statements are equivalent:

(1) R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated.
(2) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a finitely generated ideal of R_M.
(3) For every maximal ideal M of R, we have either $I_M = R_M$ or I_M is a divided proper ideal of R_M.
(4) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a principal ideal of R.
(5) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is an invertible ideal of R_M.

Remark 1.1. Suppose that R is an m-canonical ideal of R. Then dR is an m-canonical ideal of R for every nonzero nonunit d of R by Heinzer et al. (1998, Lemma 2.2(c)). Hence an integral domain admits a nonzero m-canonical ideal if and only if it admits a proper m-canonical ideal.

2. ON QUASI-LOCAL DOMAINS THAT ADMIT m-CANONICAL IDEALS

Observe that if I is an m-canonical ideal of R, then $(I : (I : R)) = R$ and hence $(I : I) = R$. In the following proposition, we show that a nonzero ideal I of R is an m-canonical ideal if and only if $(I : (I : J)) = J$ for every nonzero proper ideal J of R.

Proposition 2.1. *Let I be a nonzero ideal of R. Then the following statements are equivalent*:

(1) *I is an* m-*canonical ideal of R.*
(2) *$(I : (I : J)) = J$ for every nonzero proper ideal J of R.*

Proof. If $I = R$, then there is nothing to prove. Hence we may assume that I is a proper ideal of R. (1)$\Rightarrow$(2) No comments. (2)$\Rightarrow$(1) First, we show that $(I : I) = R$. Let $x = a/b \in (I : I)$, for some a $\in R$ and nonzero $b \in R$. Since I is an m-canonical ideal of R, we have $(I : (I : (b))) = (b)$. Since $(I : (b)) = \{i/b | i \in I\}$ and $x = a/b \in (I : I)$, we conclude that $a \in (I : (I : (b))) = (b)$. Thus $b|a$ (in R), and thus $(I : I) = R$. Hence $(I : (I : R)) = (I : I) = R$ and therefore I is an m-canonical ideal of R. □

We have the following important observation.

Proposition 2.2. *Suppose that R admits a nonzero proper* m*-canonical ideal I. Then for each maximal ideal M of R containing I, there is a $c \in R\backslash I$ such that $(I : M) = I + (c)$. In particular, if R is a quasi-local domain with maximal ideal M and $I \neq M$, then there is a $c \in M\backslash I$ such that $(I : M) = I + (c)$.*

Proof. Let M be a maximal ideal of R containing I. Since $(I : I) = R$, it is clear that $(I : M)$ is an ideal of R. Since $(I : (I : M)) = M$ and $(I : I) = R$, we conclude that there is a $c \in (I : M)\backslash I$. It is clear that $M \subset (I : I + (c))$. Once again, Since $(I : I) = R$, $(I : I + (c))$ is an ideal of R. Since M is a maximal ideal of R and $M \subset (I : I + (c)) \subset R$, the only possibilities are that either $(I : I + (c)) = M$ or $(I : I + (c)) = R$. Since I is m-canonical and $c \notin I$, the latter is ruled out since $(I : R) = I$ and $(I : (I : I + (c))) = I + (c) \neq I$. Thus $(I : I + (c)) = M$ and $(I : M) = (I : (I : I + (c))) = I + (c)$. The "in particular" statement is clear. □

It is shown in Heinzer et al. (1998, Lemma 2.2(i)) that a prime m-canonical ideal of an integral domain R is a maximal ideal of R. In the following result, we show that a proper radical m-canonical ideal of a quasi-local domain (R, M) is a maximal ideal of R.

Proposition 2.3. *Suppose that (R, M) admits proper radical* m*-canonical ideal I. Then I is a maximal ideal of R.*

Proof. Suppose that $I \neq M$. Then $(I : M) = I + (c)$ for some $c \in M\backslash I$ by Proposition 2.2. Hence $c^2 \in I$. Thus $c \in I$, a contradiction. Since (R, M) has exactly one maximal ideal, I is "the" maximal ideal of R. Hence $I = M$ is a maximal ideal of R. □

We give the following characterization of valuation domains in terms of m-canonical ideals.

Theorem 2.4. *Suppose that (R, M) is a quasi-local domain. Then R is a valuation domain if and only if R admits a proper* m*-canonical ideal I such that $(I : M)$ is a principal ideal of R.*

Proof. Suppose that (R, M) is a valuation domain. Then M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1) and hence $(M : M) = R$ is a principal ideal of R. Conversely, suppose that R admits a nonzero proper m-canonical ideal I such that $(I : M)$ is a principal ideal of R. Then $(I : M) = (d)$ for some $d \in R\backslash I$. It is clear that $R \subset (I : dM)$. Now let $x \in (I : dM)$. Then $xdM \subset I$. Since $(I : M) = (d)$ and $xdM \subset I$, we conclude that $xd \subset (d)$. Thus $x \in R$. Hence $(I : dM) = R$, and thus $dM = (I : (I : dM)) = (I : R) = I$. Let J be a nonzero ideal of R. Since $(M : (M : J)) = (dM : (dM : J))$ by Heinzer et al. (1998, Lemma 2.1), we have

$(M : (M : J)) = (dM : (dM : J)) = (I : (I : J)) = J$. Thus M is an m-canonical ideal of R, and therefore R is a valuation domain by Barucci et al. (2001, Proposition 4.1). □

Recall that a proper ideal I of an integral domain R is said to be *divided* in the sense of Dobbs (1976) and Badawi (1999) if $I \subset (c)$ for every $c \in R \backslash I$. It is clear that every proper ideal of a valuation domain is divided. We have the following result.

Corollary 2.5. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R admits a divided proper* m-*canonical ideal.*

Proof. Suppose that R is a valuation domain. Then M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1) and hence it is clear that M is a divided ideal of R. Conversely, suppose that I is a divided proper m-canonical ideal of R. Then $(I : M) = I + (c)$ for some $c \in R \backslash I$ by Proposition 2.2. Since I is divided, $(I : M) = I + (c) = (c)$. Since $(I : M)$ is a principal ideal of R, we conclude that R is a valuation domain by Theorem 2.4. □

Corollary 2.6. *Suppose that a quasi-local domain (R, M) admits a proper* m-*canonical ideal I. Then R is a valuation domain if and only if $(I : M)$ is a principal ideal of R.*

Proof. Suppose that R is a valuation domain. Then I is divided. Hence as in the proof of Corollary 2.5 we have $(I : M)$ is a principal ideal of R. The converse is clear by Theorem 2.4. □

Corollary 2.7. *Let (R, M) be a qausi-local domain. Then R is a valuation domain if and only if cM is an* m-*canonical ideal of R for some nonzero $c \in R$.*

Proof. Suppose that R is valuation domain. Then M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1). Conversely, suppose that cM is an m-canonical ideal of R for some nonzero $c \in R$. Since $(M : (M : J)) = (cM : (cM : J))$ by Heinzer et al. (1998, Lemma 2.1), for every nonzero ideal J of R we have $(M : (M : J)) = (cM : (cM : J)) = J$. □

Corollary 2.8. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if $\{cM | c$ is a nonzero element of $R\}$ is the set of all proper* m-*canonical ideals of R.*

Proof. Suppose that (R, M) is a valuation domain. Let I be a proper m-canonical ideal of R. Then $(I : M) = (c)$ for some nonzero element $c \in R$ by Corollary 2.6. Hence by an argument similar to the one just given in the proof of Theorem 2.4, we conclude that $I = cM$. □

The following lemma is needed.

Lemma 2.9. *Let $J(R)$ be the Jacobson radical of R and suppose that $J(R) \neq \{0\}$. Suppose that I is a proper ideal of $R, c_1, c_2, \ldots, c_m \in R \backslash I$ such that*

$J = I + (c_1, c_2, \ldots, c_m)$ is a finitely generated ideal of R. If L is a nonzero ideal of R which is contained in $J(R)$ and $JL = I$, then $(c_1, c_2, \ldots, c_m)L = I$.

Proof. Let L be a nonzero ideal of R which is contained in $J(R)$ and suppose that $JL = I$. Since J is a finitely generated ideal of R, we may choose $i_1, i_2, \ldots, i_n \in I$ such that $J = (i_1, i_2, \ldots, i_n, c_1, c_2, \ldots, c_m)$. Since $JL = I$, there are $d_1, d_2, \ldots, d_{n+m} \in L$ such that $d_1 i_1 + \cdots + d_n i_n + d_{n+1} c_1 + \cdots + d_{n+m} c_m = i_1$. Hence $i(1 - d_1) \in (i_2, \ldots, i_n, c_1, \ldots, c_m)L$. Since $d_1 \in J(R)$, $1 - d_1$ is a unit of R. Thus $i_1 \in (i_2, \ldots, i_n, c_1, \ldots, c_m)L$. A similar argument will show that $i_2 \in (i_3, \ldots, i_n, c_1, \ldots, c_m)L$, $i_3 \in (i_4, \ldots, i_n, c_1, \ldots, c_m)L, \ldots, i_n \in (c_1, \ldots, c_m)L$. Thus $(c_1, \ldots, c_m)L = I$. □

We have the following result.

Corollary 2.10. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if $(M : M) = R$ and R admits a proper* m*-canonical ideal I of R such that $(I : M)$ is a finitely generated ideal of R.*

Proof. Suppose that R is a valuation domain. Then it is clear that $(M : M) = R$ and M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1). Conversely, suppose that $(M : M) = R$ and R admits a proper m-canonical ideal I of R such that $(I : M)$ is a finitely generated ideal of R. Then $J = (I : M) = I + (c)$ for some $c \in R \backslash I$ by Proposition 2.2. It is clear that $R \subset (I : MJ)$. Let $x \in (I : MJ)$. Since $xMJ \subset I$, $(M : M) = R$, and $(I : J) = M$, we conclude that $xM \subset M$. Hence $x \in R$ since $(M : M) = R$. Thus $R = (I : MJ)$, and hence $MJ = (I : (I : (MJ)) = (I : R) = I$. Since $J = I + (c)$ and $MJ = I$, we conclude that $cM = I$ by Lemma 2.9. Hence $(I : M) = I + (c) = cM + (c) = (c)$ is a principal ideal of R. Thus R is a valuation domain by Theorem 2.4. □

Corollary 2.11. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R admits a proper* m*-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R and $(J : J) = R$.*

Proof. Suppose that R is a valuation domain. Then M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1) and hence the claim is clear. Conversely, suppose that R admits a proper m-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R and $(J : J) = R$. Then $J = (I : M) = I + (c)$ for some $c \in R \backslash I$ by Proposition 2.2. It is clear that $R \subset (I : JM)$. Let $x \in (I : JM)$. Since $xJM \subset I, (J : J) = R$, and $(I : M) = J$, we conclude that $xJ \subset J$. Hence $x \in R$ since $(J : J) = R$. Thus $R = (I : JM)$, and hence $JM = (I : (I : (JM)) = (I : R) = I$. Since $J = I + (c)$ and $JM = I$, we conclude that $cM = I$ by Lemma 2.9. Hence $(I : M) = I + (c) = cM + (c) = (c)$ is a principal ideal of R. Thus R is a valuation domain by Theorem 2.4. □

Corollary 2.12. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R is an integrally closed domain which admits a proper* m*-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R.*

Proof. For the converse, just observe that $(J : J) = R$ since R is integrally closed and J is a finitely generated ideal of R. Hence we are done by Corollary 2.11. □

The following is an example of a quasi-local domain (R, M) which admits a proper m-canonical ideal I such that $(I : M)$ is a finitely generated ideal of R but (R, M) is not a valuation domain.

Example 2.13. Let $V = GF(4)[[X]] = GF(4) + XGF(4)[[X]]$ is a valuation domain, and let $R = Z_2 + XGF(4)[[X]]$. Then R is a pseudo-valuation domain (pseudo-valuation domains have been defined and studied in Hedstrom and Houston, 1978) with maximal ideal $M = XGF(4)[[X]]$ which is not a valuation domain. Since $GF(4)$ is a finite algebraic extension of Z_2, R has a finitely generated m-canonical ideal I by Barucci et al. (2001, Theorem 2.16 and Theorem 3.1). Since $(I : M) = I + (c)$ for some $c \in R\backslash I$ by Proposition 2.2 and I is a finitely generated ideal of R, we conclude that $(I : M)$ is a finitely generated of R.

Remark 2.14. Observe that since (R, M) is quasi-local, the condition $(I : M)$ is a principal ideal of R in Theorem 2.4 can be replaced by $(I : M)$ is an invertible ideal of R.

Suppose that (R, M) is a valuation domain. Since every proper m-canonical ideal I of R is divided, we have $(I : M) = I + (c) = (c)$ for some $c \in R\backslash I$ by Proposition 2.2. Hence in light of the different characterizations of valuation domains above, the reader can now easily prove the following corollary.

Corollary 2.15. *Suppose that a quasi-local domain (R, M) admits a proper* m*-canonical ideal I. Then the following statements are equivalent*:

(1) *R is a valuation domain.*
(2) *I is a divided* m*-canonical ideal of R.*
(3) *$cM = I$ for some nonzero $c \in R$.*
(4) *$(I : M)$ is a principal ideal of R.*
(5) *$(I : M)$ is an invertible ideal of R.*
(6) *R is an integrally closed domain and $(I : M)$ is a finitely generated of R.*
(7) *$(M : M) = R$ and $(I : M)$ is a finitely generated of R.*
(8) *If $J = (I : M)$, then J is a finitely generated of R and $(J : J) = R$.*

Suppose that a quasi-local domain (R, M) admits a proper finitely generated m-canonical ideal I. Then $(I : M) = I + (c)$ by Proposition 2.2, and thus $(I : M)$ is a finitely generated ideal of R. Hence we have the following statements corollary.

Corollary 2.16. *Suppose that a quasi-local domain (R, M) admits a proper finitely generated* m*-canonical ideal I. Then the following statements are equivalent*:

(1) *R is a valuation domain.*
(2) *I is a divided* m*-canonical ideal of R.*
(3) *$cM = I$ for some nonzero $c \in R$.*

(4) *$(I : M)$ is a principal ideal of R.*
(5) *$(I : M)$ is an invertible ideal of R.*
(6) *R is an integrally closed domain.*
(7) *$(M : M) = R$.*
(8) *If $J = (I : M)$, then $(J : J) = R$.*

3. ON INTEGRAL DOMAINS THAT ADMIT SPECIFIC m-CANONICAL IDEALS

In this section, we investigate the behavior of integral domains that admit specific m-canonical ideals. Recall from Hedstrom and Houston (1978) that a prime ideal P of R is said to be a strongly prime ideal if, whenever $xy \in P$ with $x, y \in K$, we have $x \in P$ or $y \in P$, also recall from Badawi and Houston (2002) that an ideal I of R is called *powerful* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in R$ or $y \in R$. We recall that R is called an h-*local domain* if each nonzero ideal of R is contained in only finitely many maximal ideals of R and each nonzero prime ideal of R is contained in a unique maximal ideal of R. If I is a proper ideal of R, then $Rad(I)$ denotes the radical ideal of I in R. We start this section with the following ‘‘useful’’ lemma.

Lemma 3.1. *Let M be a maximal ideal of R and let P be a prime ideal contained in M such that $PR_M = P$. If R_M is a valuation domain, then P is a divided prime ideal of R.*

Proof. Let $r \in R\backslash P$. If R_M is a valuation domain, then $rR_M \supset P_M = P$ with $r \notin P_M$. Moreover, $rP = rP_M = P_M = P$. Thus P is a divided prime ideal of R. □

We have the following result.

Theorem 3.2. *Let I be a proper divided ideal of an integral domain R. Then*

(1) *If R is a Prüfer domain, then $Rad(I)$ is a divided prime of R.*
(2) *If R is an* h-*local Prüfer domain, then R is a valuation domain.*

Proof. (1) Suppose that R is a Prüfer domain and let M be a maximal ideal of R. Since I is divided, $I \subset M$. Moreover, $I_M \subset M$ is an ideal of R. Since R_M is a valuation domain, $Rad(I_M)$ (in R_M) is a prime ideal of the form $PR_M = P_M$ for some prime P of R minimal over I. Let $s \in P_M$. Then some power of s is in I_M. As I_M is an ideal of R and R is a Prüfer domain, s must be an element of R. Thus $P_M \subset R$ and therefore we have $P = P_M$. By Lemma 3.1, P is a divided prime of R. Thus each maximal ideal of R contains P. This implies that P is the unique minimal prime of I and therefore $Rad(I) = P$ is a divided prime of R.

(2) Suppose that R is an h-local Prüfer domain. Since R is a Prüfer domain, by (1), we conclude that $Rad(I)$ is a divided prime ideal of R, and hence $Rad(I)$ is contained in every maximal ideal of R. Since R is an h-local domain, we conclude

that $Rad(I)$ is contained in a unique maximal ideal implying that R is quasi-local and therefore a valuation domain. □

We state our main result of this section.

Theorem 3.3 (Compare with Corollary 2.5). *Suppose that R admits a divided proper* m-*canonical ideal I. Then R is a valuation domain.*

Proof. First observe that R must be an h-local domain by Heinzer et al. (1998, Proposition 2.4). Let M be a maximal ideal of R. Since I is divided, $I \subset M$. Hence I_M is a proper m-canonical ideal of R_M by Heinzer et al. (1998, Proposition 5.5). It is clear that I_M is a divided ideal of R_M since I is divided. Thus R_M is a valuation domain by Corollary 2.5. Since R_M is a valuation domain for every maximal ideal M of R, we conclude that R is a Prüfer domain and therefore an h-local Prüfer domain. Since I is divided and R is an h-local Prüfer domain, by Theorem 3.2(2), we conclude that R is a valuation domain. □

Recall from Badawi and Houston (2002) that an ideal I of R is said to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$.

Corollary 3.4. *For an integral domain R, the following statements are equivalent*:

(1) *R is a valuation domain.*
(2) *R is a root closed domain which admits a strongly primary proper* m-*canonical ideal.*

Proof. (1) ⇒ (2) Suppose that (R, M) is a strongly primary (prime) ideal of R and M is a proper m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1). Clearly, R is root closed.

(2) ⇒ (1) Let I be a strongly primary proper m-canonical ideal of R and suppose that R is root closed. Let $d \in R \backslash I$. Hence $d(i \backslash d) \in I$ for every $i \in I$. Since $d \notin I$, for every $i \in I$ we have $(i/d)^n \in I$ for some $n \geq 1$. Hence $i/d \in I$ for every $i \in I$ since R is root closed. Thus I is a divided ideal of R. Hence R is a valuation domain by Theorem 3.3. □

Corollary 3.5. *For an integral domain R, the following statements are equivalent*:

(1) *R is a one-dimensional valuation domain.*
(2) *R is a completely integrally closed domain which admits a powerful proper* m-*canonical ideal of R.*

Proof. (1) ⇒ (2) Suppose that (R, M) is a one-dimensional valuation domain. Then it is clear that R is completely integrally closed and M is a powerful ideal of R. Once again, M is an m-canonical ideal of R by Barucci et al. (2001, Proposition 4.1).

(2) $\Rightarrow$ (1) Suppose that R is completely integrally closed domain and admits a powerful proper m-canonical ideal I of R. Let $d \in R \backslash I$ and let $i \in L$. Then $(i \backslash d)^n (d \backslash i)^n i \in I$ for every $n \geq 1$. Since I is powerful, we have either $(i \backslash d)^n \in R$ or $(d \backslash i)^n \in R$. Suppose that $(d \backslash i)^n i \in R$ for every $n \geq 1$. Then since R is a completely integrally closed domain, we have $d \in (i) \subset I$, which is a contradiction since $d \notin I$. Hence $(i \backslash d)^n \in R$ for some $n \geq 1$, and thus $i \in (d)$ since R is root closed. Hence I is a divided ideal of R, and thus R is a valuation domain by Theorem 3.3. Since R is a completely integrally closed valuation domain which is not a field, R is one-dimensional. □

Theorem 3.6. *Suppose that an integrally closed domain R admits a proper* m-*canonical I such that $(I : M)$ is a finitely generated ideal of R for every maximal ideal M of R containing I. Then R is an* h-*local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated.*

Proof. Let M be a maximal ideal of R. Then I_M is an m-canonical ideal of R_M by Heinzer et al. (1998, Proposition 5.5). Suppose that $I \not\subset M$. Then $I_M = R_M$, and hence R_M is a valuation domain by Heinzer (1968, Theorem 5.1). Suppose that $I \subset M$. Then $(I_M : M_M)$ is a finitely generated ideal of R_M since $(I : M)$ is finitely generated ideal of R. Since R_M is an integrally closed domain and $(I_M : M_M)$ is a finitely generated ideal of R_M, we conclude that R_M is a valuation domain by Corollary 2.12. Hence R is a Prüfer domain, and thus R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated by Heinzer et al. (1998, Theorem 6.7). □

Heinzer et al. (1998) asked the following question which is, to my knowledge, still open: If (R, M) is an integrally closed domain that has an m-canonical ideal, does it follow that R is a valuation domain?

In case (R, M) admits a finitely generated m-canonical ideal, Barucci, Houston, Lucas, and Papick in Barucci et al. (2001, Theorem 2.1) gave a positive answer to the question above. In view of Theorem 3.6, we now give an alternative proof of Barucci et al. (2001, Theorem 2.1).

Corollary 3.7 (Barucci et al., 2001, Theorem 2.1). *Suppose that an integrally closed domain R admits a finitely generated* m-*canonical ideal. Then R is an* h-*local Prüfer domain such that each nonzero ideal of R is divisorial. In particular, if R is quasi-local, then R is a valuation domain.*

Proof. We may assume that R admits a proper finitely generated m-canonical ideal I. Let M be a maximal ideal of R containing I. Then $(I : M) = I + (c)$ for some $c \in R \backslash I$ by Proposition 2.2, and thus $(I : M)$ is a finitely generated ideal of R since I is finitely generated. Hence R is an h-local Prüfer domain by Theorem 3.6. Since I is a finitely generated ideal of R, we conclude that I is an invertible ideal of R, and thus every nonzero ideal of R is divisorial by Heinzer et al. (1998, Proposition 3.6). The "in particular" statement is clear by Corollary 2.12. □

In the light of Corollary 2.15 (Heinzer, 1968, Theorem 5.1; Heinzer et al., 1998, Proposition 5.5, Theorem 6.7), one can easily prove the following proposition.

Proposition 3.8. *Suppose that an integrally closed domain R admits a proper* m*-canonical I. Then the following statements are equivalent*:

(1) *R is an* h*-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated.*
(2) *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a finitely generated ideal of R_M.*
(3) *For every maximal ideal M of R, we have either $I_M = R_M$ or I_M is a divided proper ideal of R_M.*
(4) *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a principal ideal of R.*
(5) *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is an invertible ideal of R_M.*

We end up this paper with the following two related results. It is shown in Hedstrom and Houston (1978) that if P is a nonmaximal strongly prime ideal, then R_P is a valuation domain, also it is shown in Badawi (2000, Corollary 5) that if P is a nonmaximal strongly prime ideal, then $(P : P) = R_P$ is a valuation domain. We now give an alternative proof of Badawi (2000, Corollary 5).

Proposition 3.9 (Badawi, 2000, Corollary 5)**.** *Let P be a nonzero nonmaximal strongly prime ideal of R. Then $(P : P) = R_P$ is a valuation domain.*

Proof. It is well-known by Hedstrom and Houston (1978) that R_P is a valuation domain with maximal ideal P. Hence P is an m-canonical ideal of R_P by Barucci et al. (2001, Proposition 4.1). Hence $(P : P) = R_P$ by Heinzer et al. (1998, Lemma 2.2). □

Proposition 3.10. *Let P be a nonzero prime ideal of R. Then the following statements are equivalent*:

(1) *P is a strongly prime ideal.*
(2) *P is an* m*-canonical prime ideal of some quasi-local overring of R.*

Proof. (1) ⇒ (2) Suppose that P is a strongly prime ideal of R. If P is a maximal ideal of R, then $(P : P)$ is a valuation domain with maximal ideal P by Anderson (1983), and thus P is an m-canonical prime ideal of $(P : P)$ by Barucci et al. (2001, Proposition 4.1). Suppose that P is a nonmaximal strongly prime ideal of R. Then R_P is a valuation domain with maximal ideal P by Hedstrom and Houston (1978), and hence, once again, P is an m-canonical prime ideal of R_P by Barucci et al. (2001, Proposition 4.1).

(2) ⇒ (1) Suppose that P is an m-canonical prime ideal of a quasi-local overring B of R. Then P is a maximal ideal of B by Heinzer et al. (1998, Lemma 2.2(i)). Thus B is a valuation domain by Barucci et al. (2001, Proposition 4.1), and hence P is a strongly prime ideal of R. □

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